Quasi-Equiangular Frame (QEF) : A New Flexible Configuration of Frame

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***Abstract*—Frame theory is a powerful tool in the domain of signal processing and communication. Among its numerous configurations, the ones which have drawn much attention recently are Equiangular Tight Frame (ETF) and Grassmannian Frame. These frames both have some kind of optimality in coherence, thus bring robustness or optimal performance in applications such as digital fingerprint, erasure channels, and Compressive Sensing. However, too strict constraint on existence and construction of ETF and Grassmannian Frame became the main obstacle for widespread use. In this paper, we propose a new configuration of frame: Quasi-Equiangular Frame, as a compromise but more convenient and flexible approximation of ETF and Grassmannian Frame. We will give formal definition of Quasi-Equiangular Frame and analyze its relationship with ETF and Grassmannian frame. Furthermore, for popularity of ETF and Grassmannian frame in Compressive Sensing, we utilize the technique of random matrices to obtain asymptotical concentration estimation of the Restricted Isometry Constant (RIC) of Quasi-Equiangular Frame with respect to its key parameter.**

I. Introduction

The concept of frame theory [1] has always been a powerful tool in the domain of signal processing and communications, for its ability to deal with redundant dictionaries. The main object treated in frame theory is a sequence of functions $\{f_k\}_{k\in\mathcal{I}}$ ($\mathcal{I}$ is a countable index set) from a separable Hilbert space, which forms a frame for $\mathcal{H}$, if there exist positive constants (frame bounds) A and B such that

$$A\|f\|_2^2 \le \sum_{k\in\mathcal{I}} |\langle f, f_k\rangle|^2 \le B\|f\|_2^2 \tag{1}$$

Recently various literatures on Equiangular Tight Frame (ETF [2]) and Grassmannian Frame [3][4] are proposed. These two kinds of finite dimensional frames were warmly discussed in community of signal processing and communication. They both have the minimal coherence, i. e. the maximum correlation between different frame elements reaches the minimum. Frames with minimal coherence are preferred in applications such as digital fingerprint [5], erasure channels [6] and Compressive Sensing [7][8]. However, too strict constraint on the existence and construction of ETF and Grassmannian Frame became the main obstacle for widespread use [4][9]. Although there has been some approaches like [10] and [11] to make approximation to ETFs or Grassmannian Frames, rigorous theoretical analysis on the optimality of performance is still lack.

In this paper, we will propose a new configuration of frame, Quasi-Equiangular Frame (QEF for abbreviation), as flexible approximation of ETF and Grassmannian Frame. Its relationship with ETF and Grassmannian Frame will be exploited. The performance of QEF will be analyzed rigorously. That is, Asymptotical concentration estimation of Restricted Isometry Constant (RIC), which provide reconstruction guarantees and performance assessments for Compressive Sensing, for QEF will be given based on calculation using technique of random matrices([12][13]).

The remainder of this paper is as follows: in section II some preliminary facts about ETF and Grassmannian Frame are recalled briefly. Section III will give formal definition and main results of QTF, together with necessary analysis and discussions. In section IV the simulation result will be proposed for verification. Detailed proof for main results are attached in Appendix.

II. ETFs and Grassmannian Frames with optimal coherence

ETF and Grassmannian Frame have their optimality in coherence, i.e. the maximum correlation between different frame elements reaches the minimum. ETF are defined as follows[2]: A matrix $\Phi \in \mathbb{R}^{n\times N}$ whose columns form a frame is an Equiangular Tight Frame, if

- The columns are unit-norm or equal-norm;
- The rows are equal-norm and orthogonal, which indicates tight frames;
- The angles or correlations between distinct columns are equal.

The most remarkable property of ETF is its theoretically minimal coherence which is known as Welch Bound [14]. Here the coherence of frame $\mathbf{\Phi}$ is

$$\mu = \max_{i\neq j} |\langle \boldsymbol{\phi_i}, \boldsymbol{\phi_j}\rangle| \tag{2}$$

where $\boldsymbol{\phi_i}, \boldsymbol{\phi_j}$ are elements of frame. Furthermore, ETF is the unique frame with coherence achieving Welch Bound([14][4])

$$\mu_E := \min_{\mathbf{\Phi}} \max_{i\neq j} |\langle \boldsymbol{\phi_i}, \boldsymbol{\phi_j}\rangle| = \sqrt{\frac{N-n}{n(N-1)}} \tag{3}$$

The optimality of ETF could be demonstrated by two simple inequalities hold for general frames:

$$\|\boldsymbol{\Phi^T\Phi}\|_F^2 \geq \frac{N^2}{n} \tag{4}$$

and

$$\|\boldsymbol{\Phi^T\Phi}\|_F^2 \leq N + N(N-1)\mu^2 \tag{5}$$

In (4), the equality holds if and only if all the eigenvalues $\lambda_i(\boldsymbol{\Phi^T\Phi})$ are equal, consequence of the tight condition. On the other hand, the equality in (5) holds if and only if all the correlations between frame elements, that is $|\langle\boldsymbol{\phi_i},\boldsymbol{\phi_j}\rangle|$, are equal for all i and j, just as Equi-Angular condition. Combining (4) and (5), the optimality of ETF (3) is clear.

It can be seen that the off-diagonal elements of Gram-matrix of an ETF $\boldsymbol{\Phi^T\Phi}$'s will take only two values: $\pm\mu_E$. According to discussion on ETF from viewpoint of graph theory([8][9]). The Gram-matrix could be divided as

$$\boldsymbol{\Phi^T\Phi} = \boldsymbol{I} + \mu_E\boldsymbol{S},$$

where $\boldsymbol{I}$ denotes the identity matrix and $\boldsymbol{S}$ has zeros in the diagonal and ±1s in the off-diagonal. This $\boldsymbol{S}$ can describe the connectivity of vertices in a graph (1 for non-connectivity and -1 for connectivity, [9]).

On the other hand, Grassmannian Frame has minimal achievable coherence[4]. A unit-norm frame $\boldsymbol{\Phi} \in \mathbb{R}^{n\times N}$, is a Grassmannian Frame if

$$\mu = \min_{\boldsymbol{\Phi}}\{\mu(\boldsymbol{\Phi})\} \tag{6}$$

where the minimization is taken over all frames. It states that Grassmannian Frame achieves the minimal coherence (which may not be Welch Bound, which is not achievable for all dimension). It is obvious that the ETF is a special kind of Grassmannian Frame whose coherence achieves Welch Bound.

III. QEF and its Restricted Isometry Constants

A. Definition of QEF

First of all, we give the formal definition of QEF:

***Definition** 1:* A matrix $\Phi \in \mathbb{R}^{n\times N}$ whose columns form a frame is an QEF, if

- the columns are unit-norm or equal-norm;
- The frame correlation:

$$\mu_{i,j} := |\langle\boldsymbol{\phi_i},\boldsymbol{\phi_j}\rangle|, \quad i \neq j \tag{7}$$

  will satisfy

$$\mu_E - \varepsilon \leq \mu_{i,j} \leq \mu_E + \varepsilon, \quad i \neq j \tag{8}$$

where μ_E is Welch Bound $\sqrt{\frac{N-n}{n(N-1)}}$ (the coherence of ETF (3)).

From the definition it is clear that parameter ε denotes the deviation of frame correlations from the Welch Bound. The condition (8) places strict constraint on all the frame correlations to lie in some neighborhood centered around the Welch Bound (3) with its diameter controlled by parameter ε.

Compared with the ETF and Grassmannian Frame, QEF is essentially a different configuration of frame. Firstly, unlike ETF, QEF may not be tight. From (4), Frobenius norm of gram matrix $\boldsymbol{\Phi^T\Phi}$ of tight frame only depend on its dimensions

$$\sum_{i,j}|\langle\boldsymbol{\phi_i},\boldsymbol{\phi_j}\rangle|^2 = N^2/n \tag{9}$$

Although works on certain topics of compressive signal processing, such as construction of frames and performance analysis for Compressive Detection and Classification, adopted tightness as a natural assumption [15][16]. Its necessity is not clear and less theoretical explanation was presented to indicate the importance of tightness explicitly. Some preliminary research was conducted by ourselves to show the advantage of tight version of frame [17], but the application scenario of conclusion is limited in compressive signal detection. Furthermore, tightness places strict constraint on frame correlation which is hard to achieve. So the tightness is sacrificed in the definition of QEF. We believe that the influence on performance and scope of application of QEF is negligible for its violation of tightness. In fact, The Grassmannian Frames only demands minimal coherence, and may also not be tight. Secondly, the ε condition (8) quantitatively denotes the deviation of QEF from ETF, allowing for quantitative description of flexibility of QEF. Actually, the angle between frame elements of ETF can only take prescribed fixed values in the vector spaces with certain dimension, while that of QEF have much more freedom to take values, as is shown in Fig.1 for demonstration on the 2-dimension plane.

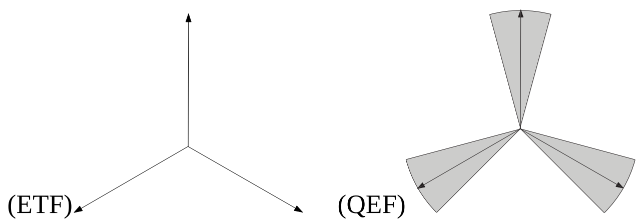

Fig. 1. Demonstration of frame angles in 2-dimension vector space for ETFs and QEFs

Fig.2 demonstrates the mutual relationship among ETF, Grassmannian frame, tight frame and QEF. The optimality of ETF is no doubt if its existence is guaranteed and it could be constructed practically. Otherwise some compromise must be taken. Grassmannian frame has minimal achievable coherence, tight frame can bring simplicity and performance improvement in some special application. QEF is more flexible and easy to be analyzed theoretically, which will be illustrated in following section.

B. The Restricted Isometry Constant of QEF

Recently the performance of ETF with its application in Compressive Sensing was discussed extensively([2][7][8][18] and reference therein). The reason of popularity of ETF is that sensing matrices with minimal coherence is preferred in sparse signal acquisition and reconstruction. Much attention was drawn on the Restricted Isometry Constants of ETF[2], which provides performance guarantees and assessments for

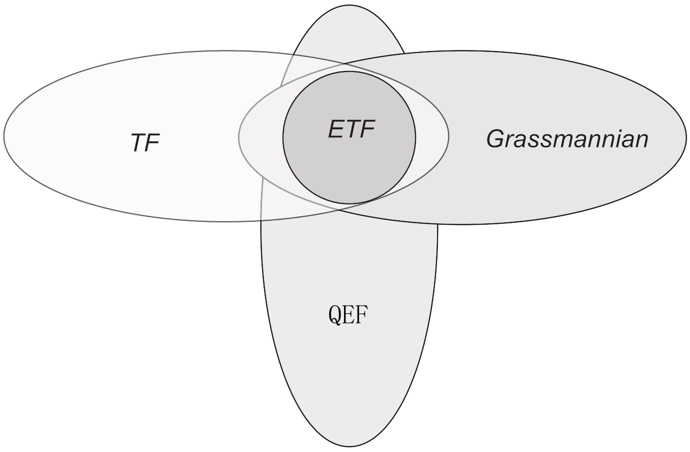

Fig. 2. Relationship between ETFs, Grassmannian Frames (Grassmannian), Tight Frames (TF) and QEFs

sparse reconstruction. Some analysis utilized tools in graph theory [7][8] to point out that a real Equiangular Tight Frame $\mathbf{\Phi}$ with coherence μ_E, has the Restricted Isometry Constant given by

$$\delta_k = (k-1)\mu_E = (k-1)\sqrt{\frac{N-n}{n(N-1)}} \tag{10}$$

for the sparsity level $k \le |\Lambda|$, where Λ denotes the set of "clique" in this corresponding connection graph of ETF. It should be noted that the "clique" means the largest set of interconnected vertices in the corresponding graph, thus $|\Lambda|$ means the size of the largest index set of the Gram-matrix $\mathbf{\Phi^T\Phi}$'s principal sub-matrix whose off-diagonal elements take the value $-\mu_E$ [2][7].

Now we will give the estimation of RIC of QEF when the key parameter ε is small. It is expected that QEF should have approximately similar RIC with ETF because it is some kind of approximation of ETF as ε tends to zero. But regarding the RIC of QEF as the same as that of ETF roughly is insufficient in general and much accurate analysis should be given to locate RIC. As well known, estimating RIC is essentially the calculation of eigenvalues of the principal sub-matrix of Gram-matrix. According to definition, Gram-matrix of QEF is perturbation of that of ETF. The existing result on estimation of eigenvalue when matrix is deterministically perturbed are mostly too crude to be used in our analysis [19]. Thus we need a new method to describe the behavior of correlations' deviation in (8) caused by ε. That is technique of randomization.

Theory of random matrices[20] has found wealthy application on diverse fields from pure mathematics and theoretical physics to communication engineering. The methods and techniques based on random matrices have been proved to be much powerful in the eigen-analysis of matrix. Much work has been done on the universal properties and sharp concentration of spectral statistics of random matrices[21][22][23]. Hence we adopted it to deal with the problem of estimating RIC of QEF. Because the Gram-matrix of QEF is deterministic, the first step we will take is randomizing it. More explicitly, the elements of Gram-matrix of QEF, which is frame correlations of QEF, are regarded as random variables uniformly distributed in the intervals formulated in (8). As the precise behavior of each correlation is unknown in practical scenario, our strategy is reasonable. The second step is to construct the concentration inequality for eigenvalues of randomized Gram-matrix of QEF. It should be mentioned that it is not very easy because the mean of elements of randomized Gram-matrix is not zero (In fact, they are μ_E), and most results concerning statistical properties of eigenvalue of random matrices took the zero mean assumption for granted [13][24]. We noticed that work of Erdős et al. [12][13] discussed the distribution of eigenvalue for non-zero mean random hermitian matrices. It was used as the basis of our derivation.

The key conclusion of Erdős et al. could be formulated as follows[12]:

***Lemma** 1:* (Erdős) Define a random hermitian matrix $\boldsymbol{A} \in \mathbb{R}^{N\times N}$

$$\boldsymbol{A} := \boldsymbol{H} + \frac{f}{N}\boldsymbol{J} \tag{11}$$

where $\boldsymbol{J}$ is a matrix whose elements are all 1's, and $\boldsymbol{H} = \{h_{i,j}\}_{i,j=1}^N$ is the random hermitian matrix with elements $h_{i,j} = h_{j,i}$ obeying independent identical distributions and

$$\mathbb{E}h_{i,j} = 0, \quad \mathbb{E}|h_{i,j}|^2 = \frac{1}{N}, \quad \mathbb{E}|h_{i,j}|^p \le \frac{C^p}{Nq^{p-2}} \tag{12}$$

where $\mathbb{E}$ denotes statistical expectation and p, q and C are some positive constants satisfying

$$3 \le p \le (\log N)^A, \quad (\log N)^B \le q \le CN^{1/2} \tag{13}$$

for some $A, B, C > 0$.

Then if the parameter f is large enough, say $f \ge 1 + c_0$ for some $c_0 > 0$, the maximum eigenvalue λ_{max} of $\boldsymbol{A}$ will satisfy

$$|\lambda_{max} - f - \frac{1}{f}| \le M(\frac{1}{f^3} + \frac{1}{f^2 q} + \frac{(\log N)^\xi}{\sqrt{N}}) \tag{14}$$

with probability

$$\mathcal{P} > 1 - e^{-\nu(\log N)^\xi} \tag{15}$$

for some constants $M > 0$, $\nu > 0$, $1 + a_0 \le \xi \le A_0 \log\log N$, (where $a_0, A_0 > 0$) and q in (13).

By utilizing the Lemma above, we obtained our main result for estimation of the RIC for the QEF.

***Theorem** 1:* Suppose an QEF $\mathbf{\Phi} = \{\phi_i\}_{i=1}^N \in \mathbb{R}^{n\times N}$ defined in Definition 1, then for ε sufficiently small, that is $\varepsilon \ll \mu_E$, and for sparsity level $k \le |\Lambda|$ where Λ denotes the set of a clique; the RIC of $\mathbf{\Phi}$ satisfies

$$|\delta_k - [(k-1)\mu_E + \frac{\sigma^2}{\mu_E}]| \le M[\frac{\sigma^4}{k\mu_E^3} + \frac{\sigma^3}{\sqrt{k}\mu_E^2 q} + (\log k)^\xi \sigma] \tag{16}$$

with probability

$$\mathcal{P} > 1 - e^{-\nu(\log k)^\xi} \tag{17}$$

for sufficiently large k and N, where μ_E, σ are:

$$\mu_E = \sqrt{\frac{N-n}{n(N-1)}}, \quad \sigma^2 = \frac{\varepsilon^2}{3} \tag{18}$$

and constants $M > 0$, $\nu > 0$, $1 + a_0 \le \xi \le A_0 \log\log k$, (where $a_0, A_0 > 0$) and $(\log k)^B \le q \le Ck^{1/2}$ as in (12).

The proof of the theorem will be given in appendix, here are some remarks.

1) For the concept of clique, we just adapt that of ETF to the QEF for the case that ε is small sufficiently, thus we argue that the clique Λ of QEF is the same as that of ETF in Theorem 1. There are further exploration of relationships between the size of clique Λ and the structure of corresponding graph in [8] and others, since it has less to do with our work, we wouldn't give further discussion on it.
2) The main result of RIC of QEF in (16) contains two part, the first part $(k-1)\mu_E$ is just the RIC of ETF (10), and the second part $\frac{\sigma^2}{\mu_E}$ is kind of deviation mainly caused by the parameter ε, where $\sigma^2 = \frac{\varepsilon^2}{3}$, thus it reasonably represented the deviation of RIC of QEF from that of ETF by ε.

IV. Simulations

In this section the accuracy of our estimation of RIC of QEF is illustrated by simulation. The dimension of the QEF is chosen to be $N = 500, n = 100 \sim 480$, and off-diagonal elements $\mu_{i,j}$ of the corresponding Gram-matrix is treated as uniform-distributed random variables, with ε chosen to be 30% of μ_E, and sparsity level k from 6 to 10. Both the result of Monte Carlo calculation and the theoretical curve from (16) was depicted in Fig. 3.

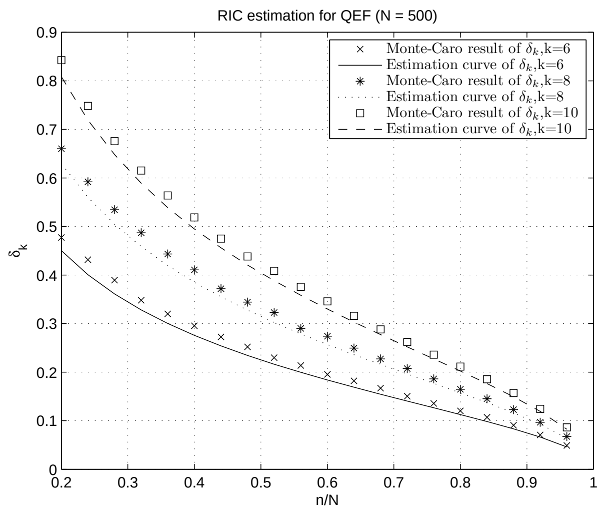

Fig. 3. Comparison between RIC estimation (16) of QEF and Monte-Caro result of RIC from simulations

It is clear that theoretical curve fits the simulation results well. Thus the validity of our result is shown intuitively.

V. Conclusion

In this paper, we proposed a new configuration of frame named QEF and discussed its relationship with other common frames including ETF and Grassmannian Frame. QTF provides a more flexible approximation of the ETF and Grassmannian Frame, its performance could be readily inquired by rigorous theoretical tools. Theory of random hermitian matrices was utilized to derive an sharp concentration estimation for the Restricted Isometry Constant (RIC) of QTF, in correspondence with the main parameter ε. Monte Carlo simulation was conducted to verify the correctness of our analytical calculation.

Appendix

The proof of Theorem 1:

Let the set Λ denotes the index set of the clique and $|\Lambda| = k$, then the Gram-matrix of the ε-ETF can be

$$\begin{aligned} \boldsymbol{\Phi}_{\boldsymbol{\Lambda}}^{\boldsymbol{T}} \boldsymbol{\Phi}_{\boldsymbol{\Lambda}} &= \boldsymbol{D}_k + \boldsymbol{S}_k + \boldsymbol{F}_k \\ &= \begin{bmatrix} 1-\mu_E & & \\ & \ddots & \\ & & 1-\mu_E \end{bmatrix} \\ &\quad + \begin{bmatrix} \mu_E & & \mu_E \\ & \ddots & \\ \mu_E & & \mu_E \end{bmatrix} + \{f_{i,j}\}_{i,j=1}^k \end{aligned} \tag{19}$$

where $f_{i,j} = f_{j,i}$ are random variables with the distribution satisfying:

$$f_{i,j} \sim p(f) = \frac{1}{2\varepsilon}, \quad f \in [-\varepsilon, \varepsilon] \tag{20}$$

Thus

$$\mathbb{E} f_{i,j} = 0, \quad \sigma^2 := \mathbb{E}|f_{i,j}|^2 = \frac{\varepsilon^2}{3} \tag{21}$$

Let

$$\boldsymbol{A}_k = \boldsymbol{S}_k + \boldsymbol{F}_k = \mu_E \boldsymbol{J} + \{f_{i,j}\}_{i,j=1}^k \tag{22}$$

and let

$$\boldsymbol{A}_k^0 = \boldsymbol{S}_k^0 + \boldsymbol{F}_k^0 = \frac{f_k^0}{k}\boldsymbol{J} + \{f_{i,j}^0\}_{i,j=1}^k \tag{23}$$

where $\boldsymbol{J}$ is a matrix whose elements are all 1's, and

$$\mathbb{E} f_{i,j}^0 = 0, \quad \mathbb{E}|f_{i,j}^0|^2 = \frac{1}{k} \tag{24}$$

thus $f_{i,j}^0 = \frac{f_{i,j}}{\sqrt{k}\sigma}$, which has the distribution as

$$f_{i,j}^0 \sim p(f) = \frac{\sqrt{k}\sigma}{2\varepsilon}, \quad f \in [-\frac{\varepsilon}{\sqrt{k}\sigma}, \frac{\varepsilon}{\sqrt{k}\sigma}] \tag{25}$$

then the moment condition in (12) will be

$$\mathbb{E}|f_{i,j}^0|^p = \frac{1+(-1)^p}{2(p+1)} \cdot (\frac{\varepsilon}{\sigma})^p \cdot \frac{1}{k^{p/2}} \le \frac{C^p}{kq^{p-2}} \tag{26}$$

for positive constant $C > 0$ and some $q \le Ck^{1/2}$.

And

$$\boldsymbol{A}_k = \sqrt{k}\sigma \boldsymbol{A}_k^0 \tag{27}$$

$$f_k^0 = k\mu_E/\sqrt{k}\sigma = \frac{\mu_E\sqrt{k}}{\sigma} \tag{28}$$

Because of $\sigma^2 = \frac{\varepsilon^2}{3} << \mu_E^2$, then $f_k^0 > 1$ is reasonable. Thus the maximum eigenvalue of $\boldsymbol{A}_k^0$ can be estimated via Lemma 1, which says:

$$|\lambda_{max}(\boldsymbol{A_k^0}) - \frac{\mu_E\sqrt{k}}{\sigma} - \frac{\sigma}{\mu_E\sqrt{k}}| \leq M(\frac{\sigma^3}{(\sqrt{k}\mu_E)^3} + \frac{\sigma^2}{(\sqrt{k}\mu_E)^2 q} + \frac{(\log k)^\xi}{\sqrt{k}}) \tag{29}$$

with probability

$$\mathcal{P} > 1 - e^{-\nu(\log N)^\xi} \tag{30}$$

for some constants $M > 0$, $\nu > 0$, $1 + a_0 \leq \xi \leq A_0 \log\log k$, (where $a_0, A_0 > 0$) and $(\log k)^B \leq q \leq Ck^{1/2}$ as in (12). Thus the maximum eigenvalue of $\boldsymbol{A}_k$ is

$$|\lambda_{max}(\boldsymbol{A}_k) - [(k\mu_E + \frac{\sigma^2}{\mu_E}]| \leq M(\frac{\sigma^4}{k\mu_E^3} + \frac{\sigma^3}{\sqrt{k}\mu_E^2 q} + (\log k)^\xi \sigma) \tag{31}$$

with probability

$$\mathcal{P} > 1 - e^{-\nu(\log N)^\xi} \tag{32}$$

Combining with the model in (19), where $\delta_k = \lambda_{max}(\boldsymbol{A}_k) - \mu_E$, the result of (16) is then clear.

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